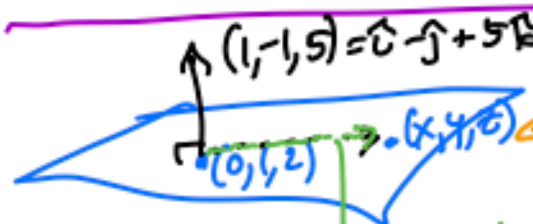
Applications of vector operations — equations of planes

Example: Find the equation of the plane that is perpendicular to $\hat{\imath} - \hat{\jmath} + 5\hat{k}$ and contains the point $(0,1,2)$.



generic point on plane

$\vec{v} = (x-0, y-1, z-2)$

(x,y,z) is on the plane

$\Leftrightarrow (1,-1,5) \cdot \vec{v} = 0$ by the given

(if and only if)

$$(1,-1,5) \cdot (x, y-1, z-2) = 0$$

$$x - y + 1 + 5z - 10 = 0$$

Equation of the plane $\boxed{x - y + 5z - 9 = 0}$

eg $y = 7$, $z = 2$

$$x - 7 + 10 - 9 = 0$$

$$x - 6 = 0 \Rightarrow x = 6$$

$\Rightarrow (6,7,2)$ is on the plane.

let $N = (1,-1,5)$ ← a normal vector for the plane

(N_1, N_2, N_3) "normal" = "perpendicular"

↑ coefficients of x, y, z in equation!

So given any equation of plane in $\mathbb{R}^3$, it looks like $$Ax + By + Cz + D = 0$$

(after simplifying)

and (A,B,C) is a normal vector for the plane.

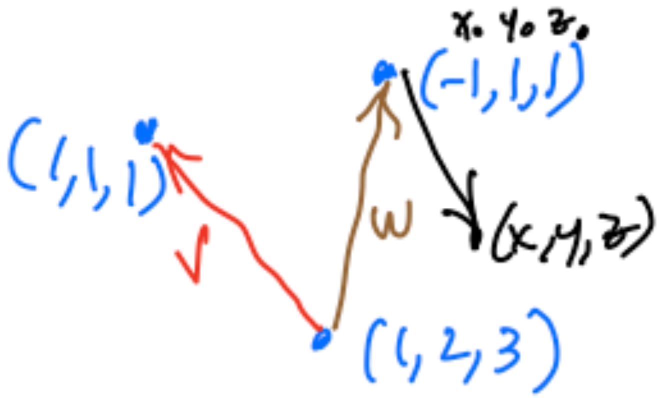
Note: There are many possible normal vectors — you could multiply by a constant to get different ones.

eg in example, $N = (1,-1,5)$

$-2(1,-1,5) = (-2,2,-10)$ is also a normal vector for the plane.

Example – Find an equation of

the plane containing $(1,1,1)$, $(-1,1,1)$, and $(1,2,3)$.



Think: Need point on plane. ✓
Need a normal vector. X

How to get a normal vector $\longrightarrow$ make two vectors & cross them.

$$v = (1,1,1) - (1,2,3) = (0,-1,-2)$$

$$w = (-1,1,1) - (1,2,3) = (-2,-1,-2)$$

$$N = v \times w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & -2 \\ -2 & -1 & -2 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -1 & -2 \\ -1 & -2 \end{vmatrix} - \hat{j}\begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix} + \hat{k}\begin{vmatrix} 0 & -1 \\ -2 & -1 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(\underbrace{0\cdot(-2) - (-2)^2}_{-4}) + \hat{k}(\underbrace{0\cdot(-1) - (-2)(-1)}_{-2})$$

$$= (0,4,-2)$$

$$\Rightarrow \quad 0(x-x_0) + 4(y-y_0) - 2(z-z_0) = 0$$

where (x_0, y_0, z_0) is on the plane.

$$\Rightarrow \quad 0(x-(-1)) + 4(y-1) - 2(z-1) = 0$$

$$\Rightarrow \quad 4y - 4 - 2z + 2 = 0$$

$$\boxed{4y - 2z - 2 = 0}$$ eqn for plane.

or $\boxed{2y - z - 1 = 0}$ (divide by 2)

For fun, let's choose $(x_0, y_0, z_0) = (1,2,3)$

$$\Rightarrow \quad N \cdot ((x-x_0), (y-y_0), (z-z_0)) = 0$$

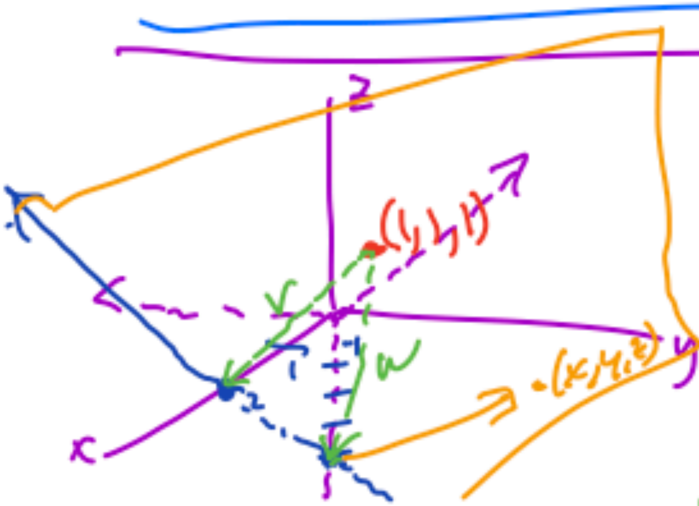
$$0(x-1) + 4(y-2) + -2(z-3) = 0$$

$$\Rightarrow \quad 4y - 8 - 2z + 6 = 0$$

$$4y - 2z - 2 = 0 \Leftrightarrow \boxed{2y - z - 1 = 0}$$

same equation! ✓

Example: Find the plane that contains the line $2x - z = 4$ in the (x,z) plane and also contains the point $(1,1,1)$.



Need for equation:
Normal vector
point on plane ✓

$v = (2,0,0) - (1,1,1)$

$w = (0,0,-4) - (1,1,1)$

$v = (1,-1,-1)$

$w = (-1,-1,-5)$

$$N = v \times w = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ -1 & -1 & -5 \end{vmatrix} = \hat{i}\begin{vmatrix} -1 & -1 \\ -1 & -5 \end{vmatrix}$$

$$- \hat{j}\begin{vmatrix} 1 & -1 \\ -1 & -5 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix}$$

$$N = \hat{i}(4) - \hat{j}(-6) + \hat{k}(-2)$$

$$= (4, 6, -2)$$

Let's use $\rightsquigarrow$ divide by 2

$(2,3,-1)$

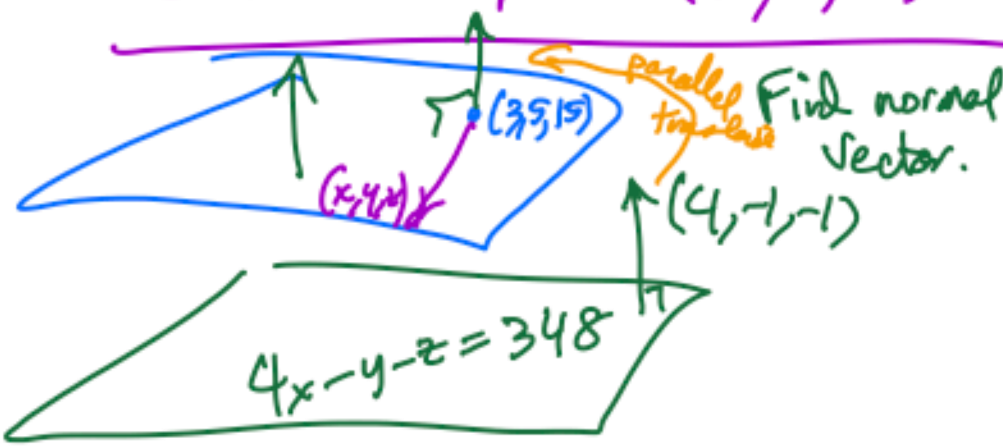
eqn. $(2,3,-1)\cdot\big((x,y,z) - (0,0,-4)\big) = 0$

$$(2,3,-1)\cdot(x,\ y,\ z+4)=0$$

$$2x+3y-(z+4)=0$$

$$\boxed{2x+3y-z-4=0}$$

<u>Example</u>. Find a plane parallel to $4x-y-z=348$ that contains the point $(3,5,15)$.



$$(4,-1,-1)\cdot(x-3,\ y-5,\ z-15)=0$$

$$4(x-3)-1(y-5)-1(z-15)=0$$

$$4x-y-z-12+5+15=0$$

$$\boxed{4x-y-z+8=0}\ \checkmark$$

Question with long answer

Find the equation of the plane containing $(1,1,0)$, $(3,0,1)$, $(1,-1,4)$.

(a) Our way: make 2 vectors, cross to get normal, use one point.

(b) Another way:

after dividing

$$Ax + By + Cz = D = 1$$

plug points in

$$A\cdot 1 + B\cdot 1 + C\cdot 0 = 1$$
$$A\cdot 3 + B\cdot 0 + C\cdot 1 = 1$$
$$A\cdot 1 + B(-1) + C(4) = 1$$

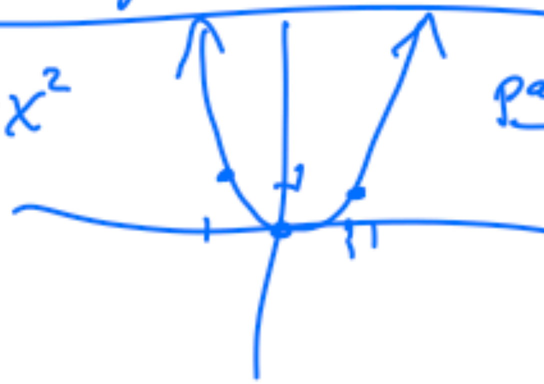
solve for A, B, C.

$$A + B = 1$$
$$3A + C = 1$$
$$A - B + 4C = 1$$

would be a long way to do it, but it would work!

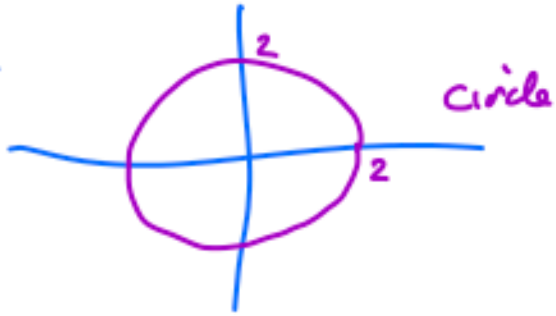
Graphs in high dimensions

$\mathbb{R}^2$: $y = x^2$

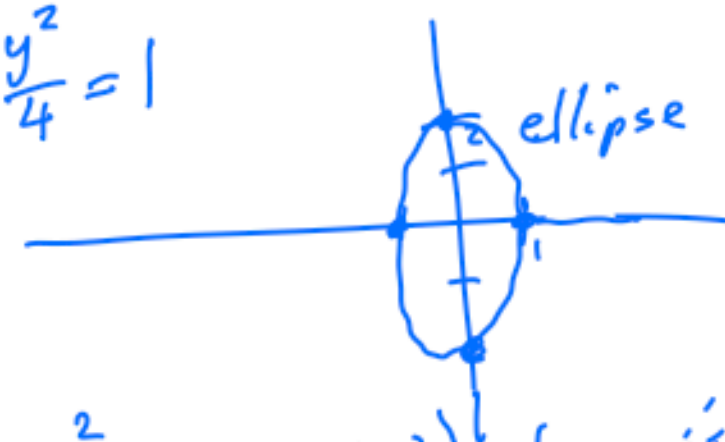


parabola

$x^2+y^2=4$



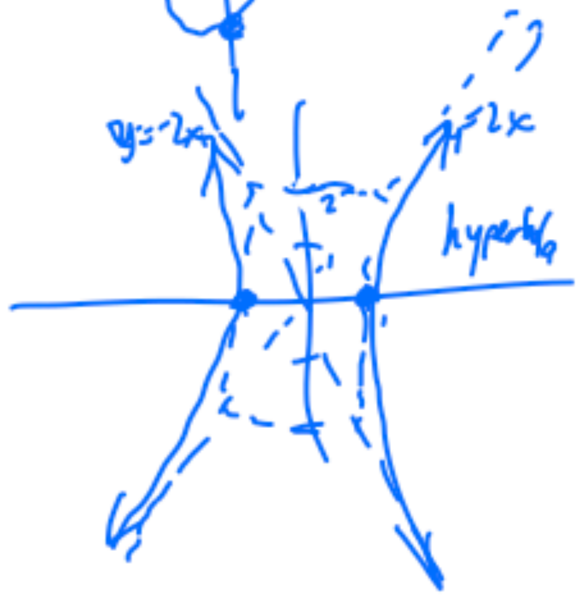
$x^2+\frac{y^2}{4}=1$



$x^2-\frac{y^2}{4}=1$

$\left(x+\frac{y}{2}\right)\left(x-\frac{y}{2}\right)=1$

$y=-2x$ $y=2x$
asymptotes



3-d graphs in (x,y,z) $\mathbb{R}^3$

If it just involves 2 variables.

$z=y^2$

X can be anything

move parabola in x direction.

$x^2+z^2=4$

y can be anything

cylinder